TRIGONOMETRICAL FORMULA

1. (A) $\sin\theta = \dfrac{1}{\csc\theta}$, $\text{cosec}\theta = \csc\theta = \dfrac{1}{\sin\theta}$, $\cos\theta = \dfrac{1}{\sec\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$,

$\tan\theta = \dfrac{1}{\cot\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(B) $\sin^2\theta + \cos^2\theta = 1, \sec^2\theta = 1 + \tan^2\theta, \csc^2\theta = 1 + \cot^2\theta$

(C) (a) $\sin ax = \dfrac{e^{iax} - e^{-iax}}{2i}$ (b) $\cos ax = \dfrac{e^{iax} + e^{-iax}}{2}$

TRIGONOMETRIC CHART:

θ	$-\theta$	$90-\theta$	$90+\theta$	$180-\theta$	$180+\theta$	$270-\theta$	$270+\theta$	$360-\theta$	$360+\theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

Notes: 1. Complementary angle (90^0, 270^0, or multiple)= Ratio becomes just opposite

(i.e *sin(90+θ)= cosθ* , , *cos(270+θ)= sinθ* , and *tan(90-θ)= cot θ* ,for the sign see quadrant table)

2. Supplementory angle (180^0, 360^0 or multiple)= Ratio remains unchanged

(i.e *sin(180+θ)= sinθ* , , *cos(360+θ)= cosθ* , for the sign see quadrant table)

3. The qydrant chart can be remember by '" **A**LL **S**TUDENTS **T**AKE **C**OFFEE" .

4. The Principal value of $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$

TRIGNOMETRIC FUNCTION OF SUM OR DIFFRENCE OF TWO ANGLES

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$, (ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$, (iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, (vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$, (viii) $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

(ix) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(x) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, (ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$, (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(i) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$, (ii) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$, (iv) $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, (ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$, (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(*MULTIPLE ANGLE*) *FORMULAE*

(i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$, (ii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(iii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

$(iv)\ \sin 3A = 3\sin A - 4\sin^3 A,\ \ (v)\ \cos 3A = 4\cos^3 A - 3\cos A,\ \ (vi)\ \tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

HALF ANGLE FORMULA

$(i)\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}},\qquad (ii)\ \tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$(iii)\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$(iv)\ 1 - \cos A = 2\sin^2(A/2),\ \ 1 + \cos A = 2\cos^2(A/2)$

(v) $\tan\theta = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$, $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta}$, $\cot\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{\sin\theta}$

HYPERBOLIC FUNCTIONS

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ $\quad \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{1-e^{-2x}}{1+e^{-2x}}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\quad \operatorname{cosech} x = \dfrac{1}{\sinh x}$, $\quad \tanh x = \dfrac{1}{\coth x} = \dfrac{\sinh x}{\cosh x}$

$\cosh(-x) = \cosh x$ $\quad \tanh(-x) = -\tanh x$

Log forms of hyperbolic functions :

$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2-1}\right\},\quad x \ge 1$	$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}$, all x	$\tanh^{-1} x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $-1 < x < 1$

Properties of Hyperbolic Functions:

$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 A = \operatorname{sech}^2 A$	$2\sinh^2 x + 1 = \cosh 2x$
$\sinh 2x = 2\cosh x \sinh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$	$2\cosh^2 x - 1 = \cosh 2x$
$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$	$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$	

Some Useful formulas: LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim\limits_{x\to\infty}\dfrac{1}{x} = 0$ $\quad$ (ii) $\lim\limits_{x\to-\infty}(1+\dfrac{1}{x})^x = e$ $\quad$ (iii) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(iv) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x}$

(v) $\lim\limits_{x\to\infty}\dfrac{e^x - 1}{x} = 1$ $\quad$ (vi) $\lim\limits_{x\to\infty}\dfrac{a^x - 1}{x} = In\ a\ \ ,a > 0$ $\quad$ (v) $\lim\limits_{x\to\infty}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

INDETERMINATE FORMS

$\boxed{\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0\times\infty\ , 0^0\ , \infty^0\ , \infty-\infty\ , 1^\infty}$

Resolve indeterminate form before using the limit by using L-hospital rule or by solving the fractions.

DIFFERENTIAL AND INTEGRAL CALCULUS

First Principle: The *derivative* of the function *f(x)* is the function *f'(x)* defined by

$$f'(x) \equiv \frac{d}{dx}[f(x)] \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

S.No	Differentiation	Integration
1	$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c,\quad n\neq -1$
2	$\frac{d}{dx}e^{ax} = ae^{ax}$	$\int e^{ax}dx = \frac{e^{ax}}{a}$
3	$\frac{d}{dx}\log_e x = \frac{1}{x}$	$\int \frac{1}{x}dx = \log x$
4	$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$	$\int a^x dx = \frac{a^x}{\log_e a}$
5	$\frac{d}{dx}\sin ax = a\cos ax$	$\int \sin ax\, dx = -\frac{\cos ax}{a}$
6	$\frac{d}{dx}\cos ax = -a\sin ax$	$\int \cos ax\, dx = \frac{\sin ax}{a}$
7	$\frac{d}{dx}\tan ax = a\sec^2 ax$	$\int \tan ax\, dx = \frac{-\log\sec ax}{a} = \frac{\log\cos ax}{a}$ $\int \sec^2 ax\, dx = \frac{\tan ax}{a}$
8	$\frac{d}{dx}\cot ax = -a\cos ec^2 ax$	$\int \cot ax\, dx = \frac{-\log\cos ecax}{a} = \frac{\log\sin ax}{a}$ $\int \cos ec^2 ax\, dx = \frac{-\cot ax}{a}$
9	$\frac{d}{dx}\sec ax\, dx = a\sec ax\tan ax$	$\int \sec ax\tan ax\, dx = \frac{\sec ax}{a}$ $\int \sec x\, dx = \log(\sec x+\tan x) = \log\ \tan(\frac{\pi}{4}+\frac{x}{2})$
10	$\frac{d}{dx}\cos ecax = -a\cos ec\, ax\cot ax$	$\int \cos ecax.\cot ax\ dx = \frac{-\cot ax}{a}$ $\int \cos ec\, x\, dx = \log(\cos ec\, x-\cot x) = \log\tan\frac{x}{2}$
11	$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x$
12	$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x$
13	$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}x$
14	$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\cot^{-1}x$

15	$\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$	$\int\frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x$
16	$\frac{d}{dx}\cos ec^{-1}x=-\frac{1}{x\sqrt{x^2-1}}$	$\int\frac{1}{x\sqrt{x^2-1}}dx=-\cos ec^{-1}x$
17	**MULTIPLICATION FORMULA** $\frac{d}{dx}f_1(x).f_2(x)=f_2(x).\frac{d}{dx}f_1(x)+f_1(x).\frac{d}{dx}f_2(x)$	**MULTIPLICATION FORMULA** $\int u.v\,dx=u\int v\,dx-\int\{\frac{d}{dx}u.\int vdx\}dx$ **Leibnitz′successive integration by Parts** $=u\int vdx-u'\int\int v\,dx^2+u''\int\int\int vdx^3...............\underbrace{\int\int\int}_{n}vdx^n$
18	**DIVISION FORMULA(Quotient Rule)** $\frac{d}{dx}(\frac{f_1}{f_2})=\frac{f_2.(\frac{d}{dx}f_1)-f_1.(\frac{d}{dx}f_2)}{(f_2)^2}$	
19	$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$	$\int\frac{1}{\sqrt{x}}dx=\int x^{-1/2}dx=\frac{x^{1/2}}{1/2}$

Some Other Formulae for Integration

$\int\frac{1}{\sqrt{a^2-x^2}}dx=\frac{1}{a}\sin^{-1}\frac{x}{a}$	$\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log(\frac{a+x}{a-x})=\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, $-a<x<a$	
$\int\frac{1}{\sqrt{a^2+x^2}}dx=\log(x+\sqrt{a^2+x^2})=\sinh^{-1}\left(\frac{x}{a}\right)$	$\int\frac{1}{\sqrt{x^2-a^2}}dx=\log(x+\sqrt{x^2-a^2})=\cosh^{-1}\left(\frac{x}{a}\right)$
$\int\sqrt{a^2-x^2}dx=\frac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}]$	
$\int\sqrt{x^2+a^2}dx=\frac{1}{2}[x\sqrt{x^2+a^2}+a^2\log(x+\sqrt{x^2+a^2})]$	$\int\sqrt{x^2-a^2}dx=\frac{1}{2}[x\sqrt{x^2-a^2}+a^2\log(x-\sqrt{x^2+a^2})]$
$\int e^{ax}.\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$	$\int e^{ax}.\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

Differentiation and Integration of Hyperbolic Functions:

$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\text{sech}\,x$	$\text{cosech}\,x$	$\coth x$
$\frac{d}{dx}f(x)$	$\cosh x$	$\sinh x$	$\sec^2 h\,x$	$-\tanh x\,\text{sech}\,x$	$-\text{cosech}\,x\coth x$	$\text{cosech}^2\,x$
$\int f(x)dx$	$\cosh x$	$\sinh x$	$\log\cos hx$	$\tan^{-1}(\sin hx)$	$\log\tan x/2$	$\log\sin hx$

Definite Integral:

1. $\int_a^b f(x)\text{dx}=\int_a^b f(y)\text{dy}=\int_a^b f(t)\text{dt}$.
2. $\int_a^b f(x)\text{dx}=-\int_b^a f(x)\text{dx}$
3. $\int_a^a f(x)\text{dx}=\int_b^b f(x)\text{dx}=\int_a^b 0\,\text{dx}=0$

4. Let $a \le c \le b$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5. (i) If $f(-x) = f(x)$ **(Even Function)** then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If $f(-x) = -f(x)$ **(Odd Function)** then $\int_{-a}^{a} f(x)\,dx = 0$

6. If $f(x)$ is periodic function, with period T i.e. $f(x+T) = f(x)$

(a) $\int_\alpha^\beta f(x)dx = \int_{\alpha+T}^{\beta+T} f(x)dx$ (b) $\int_0^\alpha f(x)dx = \int_T^{\alpha+T} f(x)dx$

Some Standard Results:

$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, $\int_0^\infty \frac{\cos x}{x}dx = \infty$, $\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$,

$\int_{-\infty}^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$, $\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, $\int_0^\infty e^{-ax}dx = \frac{1}{a}$,

$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}$,

LOGARITHM FUNCTION AND THEIR PROPERTIES

Log functions are inverses of exponential functions $y = \log_a x$ is the inverse of $y = a^x$

Properties: (1) $\log_a a = 1$ & $\log a\ 1 = 0$ & $\log_a 0 =$ not defined

(2) ***Product rule:*** $\log_a xy = \log_a x + \log_a y$ **(3) *Quotient rule:*** $\log_a \frac{x}{y} = \log_a x - \log_a y$

(4) ***Power rule :*** $\log_a x^r = r \log_a x$, ***Special case*** $\log_b b^x = x$ **(5)** $x \log y = \log y^x$

(5) Properties used to solve log equations: (a). if $a^x = a^y$, then $x = y$ (b) . if $\log_a x = \log_a y$, then $x = y$

(6) $\log_b a = \log eb \,.\, \log ac$

Natural Logarithms (1). $\ln x = \log_e x$,(2) $e^1 \approx 2.72 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots$

EXPONETIAL , TRIGNOMETRIC AND LOGARITHMIC SERIES

$(i) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\cdots\infty$

$(ii)\ e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\cdots\infty$

$(iii)\ e = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + + \frac{x^4}{4!} + \cdots\cdots\infty$

$(iv)\ e^{-1} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\cdots\infty$

$(v) a^x = 1 + \frac{x \log_e a}{1!} + \frac{(x\log_e a)^2}{2!} + \frac{(x \log_e a)3}{3!} + \cdots\infty$

(vi) *The value of 'e' lies between* 2 *and* 3 *and* $e = .71828)$

$(vii) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\infty$

$(viii)\ \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\cdots\cdots\cdots\infty$

(ix) $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots$

(x) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots$

(xi) $\tan x = \frac{x}{1} + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots\cdots\cdots\infty$

(xii) $\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots\cdots\infty$

EXISTANCE OF THE LIMIT:

Limit of a function is said to be exist if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ **= finite quantity**

Where $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(x+h)$ **and** $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(x-h)$

Rules of Operations on Limits: If $\lim_{x\to\infty} f(x)$ **and** $\lim_{x\to\infty} g(x)$ **exist , then**

(a) $\lim_{x\to\infty}[f(x)\pm g(x)] = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$ **(b)** $\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x)\cdot\lim_{x\to\infty} g(x)$

(c) $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ **if** $\lim_{x\to\infty} g(x)\neq 0$. **(d) For any constant k,**

$\lim_{x\to\infty}[kf(x)] = k\lim_{x\to\infty} f(x)$.

(e) For any positive integer ***n*****, (i)** $\lim_{x\to\infty}[f(x)]^n = [\lim_{x\to\infty} f(x)]^n$ **(ii)** $\lim_{x\to\infty}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)}$

LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim_{x\to\infty}\frac{1}{x} = 0$ **(ii)** $\lim_{x\to-\infty}(1+\frac{1}{x})^x = e$ **(iii)** $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(iv) $\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{\tan^{-1} x}{x} = \lim_{x\to 0}\frac{\sin^{-1} x}{x} = \lim_{x\to 0}\frac{\tan^{-1} x}{x}$ **(v)** $\lim_{x\to\infty}\frac{e^x-1}{x} = 1$ **(vi)** $\lim_{x\to\infty}\frac{a^x-1}{x} = In\, a \quad ,a>0$

(v) $\lim_{x\to\infty}\frac{x^n-a^n}{x-a} = na^{n-1}$

ALGEBRAIC FORMULAS –

(1) $a^2 - b^2 = (a-b)(a+b)$
(2) $(a+b)^2 = a^2 + 2ab + b^2$
(3) $a^2 + b^2 = (a-b)^2 + 2ab$
(4) $(a-b)^2 = a^2 - 2ab + b^2$
(5) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
(6) $(a-b-c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$
(7) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
(8) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
(9) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
(10) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
(11) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(12) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
(13) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$
(14) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$
(15) $a^4 - b^4 = (a-b)(a+b)(a^2 + b^2)$
(16) $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

- **If n is a natural number**, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-2}a + b^{n-1})$

- **If n is even** (n = 2k), $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-2}a - b^{n-1})$
- **If n is odd** (n = 2k + 1), $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots - b^{n-2}a + b^{n-1})$
- $(a + b + c + \ldots)^2 = a^2 + b^2 + c^2 + \ldots + 2(ab + ac + bc + \ldots$
- **Laws of Exponents** $(a^m)(a^n) = a^{m+n}$, $(ab)^m = a^m b^m$, $(a^m)^n = a^{mn}$
- **Fractional Exponents** $a^0 = 1$ $a^m / a^n = a^{m-n}$, $a^m = 1/a^{-m}$, $a^{-m} = 1/a^m$

ROOTS OF QUADRATIC EQUATION

- For a quadratic equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$,
  $\Delta = \dfrac{-(b) \pm \sqrt{b^2 - 4ac}}{2a}$ is called the discrimination.
- For real and distinct roots, $\Delta > 0$
- For real and coincident roots, $\Delta = 0$
- For non-real roots, $\Delta < 0$
- If α and β are the two roots of the equation ax2 + bx + c then, $\alpha + \beta = (-b / a)$ and $\alpha \times \beta = (c / a)$.
- If the roots of a quadratic equation are α and β, the equation will be $(x - \alpha)(x - \beta) = 0$

FACTORIALS

- $n! = (1).(2).(3)\ldots..(n - 1).n$ **OR** $n! = n(n - 1)! = n(n - 1)(n - 2)! = \ldots.$
- $0! = 1$

SOLUTION OF QUADRATIC EQUATION:

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$. then its solution $x = \dfrac{-(b) \pm \sqrt{b^2 - 4ac}}{2a}$

THE GENERAL BINOMIAL EXPANSION

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \text{ where } {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

OR

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots\ldots\ldots\ldots + {}^nC_na^{n-n}b^n$$

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3}a^{n-3}b^3 + \ldots\ldots\ldots\ldots + b^n$$

OR

We can write individual expressions for each of the binomial coefficients...

MAKING FACTOR (PARTIAL FRACTIONS):

1. $\dfrac{f(x)}{(x-a)(x-b)(x-c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$ [All Linear Factors]

2. $\dfrac{f(x)}{(x-a)^2(x\pm c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x\pm d)}$ [One or more factor are whole square]

3. $\dfrac{f(x)}{(ax^2+bx+c)(x\pm d)} = \dfrac{Ax+B}{(ax^2+bx+c)} + \dfrac{C}{(x\pm d)}$ [One or more factor is quadratic equation]

NUMERICAL ANALYSIS

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS:

Algebraic and Transcendental equation: An equation of the form $f(x) = a_0x^n + a^1x^{n-1} + \ldots\ldots + an = 0$ is called an algebraic equation.

An equation consisting trigonometric function, exponential function, logarithmic function etc. is called a transcendental equation.

Solution of Algebraic and Transcendental equations:

Bisection Method:

(i) Find the negative and positive values of the function at two different points ,
(ii) Say *f(a)*= - *Ve* and *f(b)*= +*ve* (Then Root lies b/w a and b)
(iii) Take $a=x_0$ and $b=x_1$
(iv) Find $x_2= x_0 + x_1 /2$
(v) Find $f(x_2)$
(vi) If $f(x_2)$=+ve then root lies b/w $a=x_0$ and x_2
If $f(x_2)$= - ve then root lies b/w $b=x_1$ and x_2 , repeat procedure from (iii)

Regula Falsi Method (Or Method of false positions)

1. find the negative and positive values of the function at two different points
2. say *f(a)*= -Ve and *f(b)*= +ve(Then Root lies b/w *a* and *b*)
3. let $a=x_0$ and $b=x_1$
4. Find $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$
5. find $f(x_2)$
6. If $f(x_2)$=+ve then root lies b/w $a=x_0$ and x_2
7. If $f(x_2)$= - ve then root lies b/w $b=x_1$ and x_2 , repeat procedure from (2)

Newton Rap son's Method:

1. Find the negative and positive values of the function at two different points
2. Say *f(a)*= -Ve and *f(b)*= +ve
3. If | *f(a* |< | *f(b)*| **(Numerical Value, without sign) ,** then take $a= x_0$
4. Find $x_{n+1}= x_n - \dfrac{f(x_n)}{f'(x_n)}$, Provided $f'(x_n)$ exist
5. Find net approximations using (2)

Secant Method:

This method is same as the *Regula Falsi Method, but in this method* ***we not need to check the +ve and –ve sign in each step***. We can use general formula $x_{n+2} = \dfrac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$

DIFFERENCE OPERATORS:

(1) Shifting Operator: $E\,f(x) = f(x+h)$, $E^2 f(x) = f(x+2h)$, ….. $E^n f(x) = f(x+nh)$, or $E\,y_x = y_{x+h}$, $E^n y_x = y_{x+nh}$,

(2) *Forward difference operator:* $\Delta f(x) = f(x+h) - f(x)$ or $\Delta y_x = y_{x+h} - y_x$

(3) *Backward difference operator :* $\nabla f(x) = f(x) - f(x-h)$ or $\nabla y_x = y_x - y_{x-h}$

(4) *Averaging operator :* $\mu\, f(x) = \dfrac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}$

(5) *Central difference operator* $= \delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

(6) $E = e^{hD}$ [Hint : use Taylor Series $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x)$……. Then we get $Ef(x) = e^{hD} f(x)$]

$$\Delta = E - 1$$
$$\nabla = 1 - E^{-1}$$
$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$
$$\delta = E^{1/2} - E^{-1/2}$$

FIND MISSING TERMS:

If there are n missing terms/ data in the given table then $\Delta^{n-1} y_x = 0$ or $\Delta^{n-1} f(x) = 0$, use $\Delta = E - 1$ and and expand the series using binomial theorem $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots\dots\dots\dots + {}^nC_n a^{n-n} b^n$

OR

$$(a+b)^n = a^n + n a^{n-1} b^1 + \frac{n(n-1)}{2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3} a^{n-3} b^3 + \dots\dots\dots\dots + b^n$$

i.e

$$(E-1)^5 y_x = \left(E^5 + 5E^4(-1)^1 + \frac{5.(5-1)}{2} E^3(-1)^2 + \frac{5.(5-1)(5-2)}{3} E^2(-1)^3 + \frac{5.(5-1)(5-2)(5-3)}{4} E^1(-1)^4 + \frac{5.(5-1)(5-2)(5-3)(5-4)}{5} E^0(-1)^5 \right) y_x = 0$$

or $y_{x+5} - 5y_{x+4} + 10y_{x+3} - 20y_{x+2} + 10y_{x+1} - y_x = 0$ (Since $E^n y_x = y_{x+n}$), Put x= 0,1….. and solve the algebraic eq.s

FACTORIAL POLYNOMIALS:

The factorial polynomial is the continued product of the factors in which the first factor is x and the successive factors decrease by a constant h and is denoted by $x^{(n)}$. Where $x^{(n)} = x(x-h)(x-2h)\dots\dots\dots\dots\{x-(n-1)h\}$

i.e. $x^{(1)} = x$, $x^{(2)} = x(x-1)$, $x^{(3)} = x(x-1)(x-2)$……

$$\Delta x^{(n)} = n x^{(n-1)},\ \Delta^2 x^{(n)} = n(n-1) x^{(n-2)} \dots\dots \text{ and } \frac{1}{\Delta} x^{(n)} = \frac{x^{(n+1)}}{n+1},\ \frac{1}{\Delta^2} x^{(n)} = \iint x^{(n)} dx = \frac{x^{(n+2)}}{(n+1)(n+2)}$$

INTERPOLATION:

Interpolation is the process to find the values of y for any intermediate value of x between the interval.

Extrapolation is the process to find the values of y for any value of x outside the interval.

INTERPOLATION WITH EQUAL INTERVALS:

1. Gregory- Newton's Forward difference interpolation formula: When required value of y=f(x) is near to the top then use forward difference interpolation formulae.

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots + \frac{p(p-1)\ldots\ldots\ldots(p-(p-1))}{n!}\Delta^n y_0 \ldots\ldots\ldots\ldots + \left[Where\ p = \frac{(x-x_0)}{h} \right]$$

2. Gregory- Newton's Backward difference interpolation formula:[*When required value of y=f(x) is near to the bottom i.e. x_n , then use backward difference interpolation formulae. It is also used for extrapolating values of y for x , when x is slightly grater than x_n :*

$$y = f(x) = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots\ldots + \frac{p(p+1)\ldots\ldots\ldots(p+(p-1))}{n!}\nabla^n y_n \ldots\ldots\ldots\ldots + \left[Where\ p = \frac{(x-x_n)}{h} \right]$$

Central difference interpolation formulas: [When required value of y=f(x) is near to the middle , then use central difference interpolation formulae.]

1. Gauss forward difference interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \ldots\ldots\ldots(0 < p < 1)$$

2. Gauss Backward difference interpolation formula:

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \ldots\ldots\ldots(-1 < p < 0)$$

Gauss Backward	Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$
y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
Gauss Forward	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$

3. Sterling Formula: { Sterling formula is the mean of gauss forward and back ward formula}

$$y = f(x) = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} \ldots,\ (\frac{-1}{4} < p < \frac{1}{4})$$

4. Bessel's Formula :{Shift the origin to 1 by replacing p by (p-1) & add 1 to each argument 0,-1,-2….in gauss backward formulas and , take mean of gauss forward formula and revised backward formula}

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{p(p-1)(p-1/2)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots..,\ (\frac{1}{4} < p < \frac{3}{4})$$

INTERPOLATION WITH UNEQUAL INTERVALS:

Divided difference: $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, $f[x_0, x_1] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

1. **Newton's Divided difference interpolation formula:**

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ldots\ldots\ldots$$

1. **Lagrange's Interpolation formula:**

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1) + \ldots\ldots\ldots\ldots$$

2. **Inverse Lagrange's Interpolation formula:**

$$x = f^{-1}(y) = \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}(x_0) + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}(x_1) + \ldots\ldots\ldots\ldots$$

NUMERICAL DIFFERENTIATION:

1. Differentiate Newton's forward interpolation formula with respect to "*p*" we get following

Newton's forward difference formula:

$$f'(x) = f'(a+ph) = \frac{1}{h}\left[\Delta f(a) + \frac{2p-1}{2!}\Delta^2 f(a) + \frac{3p^2-6p+2}{3!}\Delta^3 f(a) + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 f(a) + \ldots\ldots\ldots\right]$$

$$f''(x) = f''(a+ph) = \frac{1}{h^2}\left[\Delta^2 f(a) + (p-1)\Delta^3 f(a) + \frac{6p^2-18p+11}{12}\Delta^4 f(a) + \ldots\ldots\ldots\right]$$

When $x=x_0$ then $p=x-x_0/h = 0$ hence these formulae reduce to

$$f'(x) = f'(a) = \frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \frac{1}{4}\Delta^4 f(a) + \ldots\ldots\ldots\right]$$

Newton's Backward difference formula:

$$f'(x) = f'(a+ph) = \frac{1}{h}\left[\nabla f(x_n) + \frac{2p+1}{2!}\nabla^2 f(x_n) + \frac{3p^2+6p+2}{6}\nabla^3 f(x_n) + \frac{4p^3-18p^2+22p-6}{4!}\nabla^4 f(x_n) + \ldots\ldots\ldots\right]$$

$$f''(x) = f''(a+ph) = \frac{1}{h^2}\left[+\nabla^2 f(x_n) + (p+1)\nabla^3 f(x_n) + \frac{6p^2+18p+11}{12}\nabla^4 f(x_n) + \ldots\ldots\ldots\right]$$

When $x=x_0$ then $p=x-x_0/h = 0$ hence these formulae reduce to

$$f'(x) = f'(a) = \frac{1}{h}\left[\nabla f(x_n) + \frac{1}{2}\nabla^2 f(x_n) + \frac{1}{3}\nabla^3 f(x_n) - \frac{1}{4}\nabla^4 f(x_n) + \ldots\ldots\ldots\right]$$

NUMERICAL INTEGRATION:

Area Bounded between the limits x_n and x_0 is called integration b/w the limits x_n and x_0.

(1) **Trapezoidal Rule:** $\displaystyle\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots\ldots + y_{n-1}\right)\right]$

(2) **Simpson's 1/3 Rule: [4(odd)+2(even)]** *[divide the interval in multiple of 2]*

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+...y_{n-1})+2(y_2+y_4+............+y_{n-2})]$$

(3) **Simpson's 3/8 Rule: [3(1,2,4,5,7……left multiple of 3)+ 2(3,6,9………multiple of 3)]**
[divide the interval in multiple of 3]

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+.....+y_{n-2}+y_{n-1})+2(y_3+y_6+............+y_{n-3})]$$

(4) **Weddle Rule:[1,5,1,6,1,5,1]** *[divide the interval in multiple of 6]*

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{10}[(y_0+5y_1+y_2+6y_3+y_4+5y_5.+y_6)+(y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}.+y_{12})+......]$$

Note: (1) n -ordinate means $n= n-1$ in $h=(x-x_0)/n$ (2) n- equidistance intervals means $n=n-1$ (3) n -equal parts means $n=n$

SOLUTION OF ALGEBRAIC SIMULTANEOUS LINEAR EQUATIONS:

Linear Algebraic Equations: Let system of linear equations is:

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3$$

1. DIRECT METHODS:

(i) Gauss Elimination method (Method of Pivoting) :

In essence, we wish to eliminate unknowns from the equations by a sequence of algebraic steps.

Let augmented matrix **[A:b]=** $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

Normalization (i) Let $a_1 \neq 0$. Then by $27x+6y-z=85$, $6x+15y+2z=72$, $x+y+54z=110$

and $R_{31}(\frac{-a_3}{a_1}) \Rightarrow R_3 = R_3 - \frac{a_3}{a_1}R_1$,we get $\approx \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$ here a1 is called **pivoting element**.

Reduction : Now take b_2' ($\neq 0$) as the pivoting element , and use $R_{32}(\frac{-b_3'}{b_2'}) \Rightarrow R_3 = R_3 - \frac{b_3'}{b_2'}R_2$

We get $\approx \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$ after solving this matrix by back substitution we get required results.

Note: This method fails if a_1, b_2' or c_3'' becomes zero. In such cases by inter changing the rows we can get the non zero pivots.

(ii) Gauss Jordan Method:

It is a variation of Gauss elimination. The differences are:

- When an unknown is eliminated from an equation, it is also eliminated from all other equations.

- All rows are **normalized by dividing them by their pivot element.**

Hence, the elimination step results in an identity matrix rather than a triangular matrix. Back substitution is, therefore, not necessary.

All the techniques developed for Gauss elimination are still valid for Gauss-Jordan elimination.

GAUSS-JORDAN ELIMINATION:

1. Get a 1 in upper left corner (by row ops 1 and/or 2)

2. Get 0's everywhere else in its column (by row op 3)

3. Mentally delete row 1 and column 1. What remains is a smaller **submatrix**.

4. Get 1 in upper left-hand corner of the *sub matrix*.

5. Get 0's everywhere else in its column for *all rows* in the matrix (not just the submatrix).

6. Mentally delete row 1 and column 1 of the submatrix, forming an even smaller submatrix.

7. Repeat 4, 5, 6 until you can go no further.

8. The matrix will now be in **reduced row-echelon form** (RREF), or just **reduced form**.

6. Re-write the system in natural form.

7. State the solution.

A. If you get a row of all zeros, use row op 1 to make it the last row

B. If you get a row with all zeros to the left of the line, and a non-zero on the right, STOP (no solution).

(iii)LU Factorization Method(or Crout's Method , or Choleskey's Method)

For a nonsingular matrix [*A*] on which one can successfully conduct the Naïve Gauss elimination forward elimination steps, one can always write it as

Step –I TAKE [A]=[L][U]

Where : [L]= Lower triangular matrix with unit diagonal $=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$,

[U] = Upper triangular matrix= $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Step –II : Take [L][Z]=[b]

Step-III . [U][X]=[Z] Where Z=[z_1, z_2, z_3] **Step-IV :** Use back Substitution to find values of *x, y, z*

ITRATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATION:

(i) ***Jacobi Method* :** Let system of equations is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

Solve each equation for one variable:

For first equation $|a_{11}| > |a_{11}| + |a_{12}| \dots\dots + |a_{1n}|$, For Second equation $|a_{22}| > |a_{21}| + |a_{23}| \dots\dots + |a_{2n}|$ ……

$$x_1 = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n)\right]$$
$$x_2 = \frac{1}{a_{22}}\left[b_2 - (a_{21}x1 + a_{23}x_3 + \dots + a_{2n}x_n)\right]$$
$$\dots\dots\dots$$
$$.x_n = \frac{1}{a_{nn}}\left[b_n - (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{n,n-1}x_{n-1})\right]$$

The Iteration formulas are

$$\boxed{\begin{aligned} x_1^{(i+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(i)} + a_{13}x_3^{(i)} + \dots + a_{1n}x_n^{(i)}\right)\right] \\ x_2^{(i+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(i)} + a_{23}x_3^{(i)} + \dots + a_{2n}x_n^{(i)}\right)\right] \\ &\dots\dots\dots \\ x_n^{(i+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(i)} + a_{n2}x_2^{(i)} + \dots + a_{n,n-1}x_{n-1}^{(i)}\right)\right] \end{aligned}}$$

Gauss-Seidel Method:

In most cases using the newest values on the right-hand side equations will provide better estimates of the next value. If this is done, then we are using the Gauss-Seidel Method:

The Iteration formulas are:

$$\boxed{\begin{aligned} x_1^{(i+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(i)} + a_{13}x_3^{(i)} + \dots + a_{1n}x_n^{(i)}\right)\right] \\ x_2^{(i+1)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(i+1)} + a_{23}x_3^{(i)} + \dots + a_{2n}x_n^{(i)}\right)\right] \\ &\dots\dots\dots \\ x_n^{(i+1)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(i+1)} + a_{n2}x_2^{(i+1)} + \dots + a_{n,n-1}x_{n-1}^{(i)}\right)\right] \end{aligned}}$$

Note: 1. ***Why use Jacobi ? Ans:*** Because you can separate the n-equations into n independent tasks; it is very well suited to computers with parallel processors.

2. *If either method converges, Gauss-Seidel converges faster than Jacobi.*

RELAXATION METHOD

Solve the system of linear algebraic equation using relaxation Method

$$10x-2y-3z=205,\ -2x+10y-2z=154\ ,\ -2x-y+10z=120$$

Solution: The Residuals are

$$R_x=205-10x+2y+3z \ldots\ldots..(1)$$
$$R_y=154+2x-10y+2z \ldots\ldots .(2)$$
$$R_z=120+2x+y-10z \ldots\ldots..(3)$$

Operation Table

	δRx	δRy	δRz	
δx	-10	2	2	Diff eq.(1),(2),(3) w.r.t. *"x"* Respectively
δy	2	-10	1	Diff eq.(1),(2),(3) w.r.t. *"y"* Respectively
δz	3	2	-10	Diff eq.(1),(2),(3) w.r.t. *"z"* Respectively

In this method we reduce(minimize) residuals by giving increments to the variables. The process stop when residuals become "0" or near to "0".

Residuals Table

Operations	Values	Rx	Ry	Rz
Initially we put	$x=y=z=0$	205	154	120
Since the max. residual is 205 in Rx , hence we take approximate value of $\delta x=\frac{Rx}{\lvert a_1\rvert}$	$\boxed{\delta x=\frac{205}{\lvert -10\rvert}=20.5\approx 20}$ Put this value in eq.(1),(2),(3) keeping y, z constant	Since the residual i.e.205 , or eq. is $R_x=205-10x+2y+3z$ (put the value of δx=20, keeping y,z constant) $\boxed{205-10(20)=5}$	Since the residual i.e.154 , or eq. is $R_y=154+2x-10y+2z$ (put the value of δx=20, keeping y,z $\boxed{154+2(20)=194}$	Since the residual i.e.120 , or eq. is $R_z=120+2x+y-10z$ (put the value of δx=20, keeping y,z constant) $\boxed{120+2(20)=160}$
Since the max. residual is 194 in Ry , hence we take approximate value of $\delta y=\frac{Ry}{\lvert b_2\rvert}$	$\boxed{\delta y=\frac{194}{\lvert -10\rvert}=19.4\approx 19}$ Put this value in eq.(1),(2),(3) keeping x, z constant	Since new residual i.e.5 , or New eq. becomes $R_x=5-10x+2y+3z$) (put the value of δy, keeping x,z constant) $\boxed{5+2(19)=43}$	(use new residual i.e.194 , or New eq. becomes $R_y=194+2x-10y+2z$ put the value of δy, keeping x,z constant) $\boxed{194-10(19)=4}$	(use new residual i.e.160 , or New eq. becomes $R_z=160+2x+y-10z$ put the value of δy, keeping x,z constant) $\boxed{160+19=179}$
Since the max. residual is 179 in Rz , hence we take approximate value of $\delta z=\frac{Rz}{\lvert c_3\rvert}$	$\boxed{\delta z=\frac{179}{\lvert -10\rvert}=17.9\approx 18}$ Put this value in eq.(1),(2),(3) keeping x, z constant	(use new residual i.e.5 , or New eq. becomes $R_x=43-10x+2y+3z$ Put the value of δz=18 , keeping x, y constant $\boxed{43+3(18)=97}$	use new residual i.e.5 , or New eq. becomes $R_y=4+2x-10y+2z$ Put the value of δz=18 , keeping x, y constant $\boxed{4+2(18)=40}$	use new residual i.e.179 , or New eq. becomes $R_z=179+2x+y-10z$ Put the value of δz=18 , keeping x, y constant $\boxed{179-10(18)=-1}$
Since the max. residual is 97 in	$\boxed{\delta x=\frac{97}{\lvert -10\rvert}=9.7 \approx 10}$	(use new residual i.e.97 , or	use new residual i.e.40 , or	use new residual i.e.-1 , or

Rx , hence we take approximate value of $\delta x = \frac{Rx}{\lvert a_1 \rvert}$	Put this value in eq.(1),(2),(3) keeping *y, z* constant	New eq. becomes $R_x = 97 - 10x + 2y + 3z$ Put the value of δx=10 , keeping *y,z* constant $\boxed{97 - 10(10) = -3}$	New eq. becomes $R_y = 40 + 2x - 10y + 2z$ Put the value of δx=10 , keeping *y,z* constant $\boxed{40 + 2(10) = 60}$	New eq. becomes $R_z = -1 + 2x + y - 10z$ Put the value of δx=10 , keeping *y,z* constant $\boxed{-1 + 2(10) = 19}$
Since the max. residual is 60 in Ry , hence we take approximate value of $\delta y = \frac{Ry}{\lvert b_2 \rvert}$	$\boxed{\delta y = \frac{60}{\lvert -10 \rvert} = 6}$ Put this value in eq.(1),(2),(3) keeping *x, z* constant	(use new residual i.e.-3 , or New eq. becomes $R_x = -3 - 10x + 2y + 3z$ Put the value of δy=6 , keeping *x, z* constant $\boxed{-3 + 2(6) = 9}$	use new residual i.e.60 , or New eq. becomes $R_y = 60 + 2x - 10y + 2z$ Put the value of δy=6, keeping *x, z* constant $\boxed{60 - 10(6) = 0}$	use new residual i.e. 19 , or New eq. becomes $R_z = 19 + 2x + y - 10z$ Put the value of δy=6, keeping *x, z* constant $\boxed{19 + 6 = 25}$
Since the max. residual is 25 in Rz , hence we take approximate value of $\delta z = \frac{Rz}{\lvert c_3 \rvert}$	$\boxed{\delta z = \frac{25}{\lvert -10 \rvert} = 2.5 \approx 2}$ Put this value in eq.(1),(2),(3) keeping *x, y* constant	(use new residual i.e9 , or New eq. becomes $R_x = 9 - 10x + 2y + 3z$ Put the value of δz=2 , keeping *x, y* constant $\boxed{9 + 3(2) = 15}$	use new residual i.e.0 , or New eq. becomes $R_y = 0 + 2x - 10y + 2z$ Put the value of δz=2 , keeping *x, y* constant $\boxed{0 + 2(2) = 4}$	use new residual i.e. 25 , or New eq. becomes $R_z = 25 + 2x + y - 10z$ Put the value of δz=2 , keeping *x, y* constant $\boxed{25 - 10(2) = 5}$
Similarly	δx=2	-5	8	9
	δz=1	-2	10	-1
	δy=1	0	0	0

Since all the residuals are zero(or may be near equal to zero) hence we stop the process , also x

$$x = \sum \delta x = 20 + 10 + 2 = 32 \text{ , } y = \sum \delta y = 19 + 6 + 1 = 26 \text{ , } z = \sum \delta z = 18 + 2 + 1 = 21$$

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

TAYLOR SERIES METHOD:

Consider the initial-value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. then its solution $y(x) = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + ...$

PICARD METHOD (Picard's Method of Successive Approximation)

Let us consider the initial-value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

then $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$, $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$,……….. $y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)dx$

EULER'S METHOD : Given the initial-value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ defined on the interval $x_0 \le x \le x_0 + h$,

then at $\mathbf{x_1 = x_0 + h}$, $x_2 = x_1 + h$…..the **approximate value** of $y(x_0+h)$, denoted by y_1, is given by

$$y_1 = y_0 + h[f(x_0, y_0)] \text{ , } \quad y_2 = y_1 + h[f(x_1, y_1)] \quad \ldots\ldots\ldots \quad y_n = y_{n-1} + h[f(x_{n-1}, y_{n-1})]$$

Modified Euler's method

First find y_1 , using Euler's method and then apply modify formula $y_1 = y_0 + h[f(x_0, y_0) + f(x_1, y_1)]$ where $x_1 = x_0 + h$ and y_1 is from Euler formula. Similarly Find required approximations.

Alternatively: Find $y_1 = y_0 + h[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)]$, $y_2 = y_1 + h[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)]$

$$y_{n+1} = y_n + h[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)]$$

RUNGE-KUTTA METHOD: Given the initial-value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Find $k_1 = f(x_0, y_0)$, $k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2})$, $k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{k_2}{2})$, $k_4 = f(x_0 + h, y_0 + k_3)$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Milne's Predictor-corrector method: The third-order equations for predictor and corrector are

Let differential equation is $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

(1) First find four starting values of y values by any previous method(Taylor series, Euler's method, Picard Method , etc.).Find y_0, y_1, y_2, y_3 for x_0 , x_1, x_2, x_3 respectively.

(2) Find y'_0, y'_1, y'_2, y'_3 (from $\frac{dy}{dx} = f(x, y)$)

(3) Find y_4 using **Milne's Predictor formula** $y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$ and find $y_4' = f(x_4, y_4)$

(4) Use **Milne's Corrector formula** and find $y_4^{(1)} = y_2 + \frac{h}{3}[2y_2' + 4y_3' + y_4']$

again $y_4^{(2)} = y_2 + \frac{h}{3}[2y_2' + 4y_3' + y_4'^{(1)}]$, $y_4^{(3)} = y_2 + \frac{h}{3}[2y_2' + 4y_3' + y_4'^{(2)}]$ ……

Continue this process, when two consecutive approximations are equal at desire places ***For This method min. value of n=4**

ADAMS-BASHFORTH-MOULTON METHOD FOR O.D.E.'S

[1]. Let differential equation is $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

[2]. First find four starting values of y values by any previous method(Taylor series, Euler's method, Picard Method , etc.).Find y_0, y_1, y_2, y_3 for x_0 , x_1, x_2, x_3 respectively.

[3]. Find y'_0, y'_1, y'_2, y'_3 (from $\frac{dy}{dx} = f(x, y)$)

[4]. Tthen apply the predictor formula $y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$,

[5]. The corrector formula $y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$

Fourier Transform

If $f(x)$ is the function defined in the interval $(-\infty , \infty)$, uniformly continuous in the finite intervals and $\int_{-\infty}^{\infty} |f(x)| dx$ converges then the Fourier transform of a one-dimensional function f(x) is defined as

$$\Im[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx \quad .$$

The inverse transform $\Im^{-1}$ is defined as $\Im^{-1}[F(s)] = f(x) = \int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$, Where s is a parameter.

It may be represented by $\Im[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$ and

$$\Im^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

and $\Im[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{2\pi isx}\,dx$ and $\Im^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-2\pi isx}\,ds$

Remark: 1. Since $|x| = \begin{cases} x & ,x>0 \\ -x & ,x<0 \end{cases}$ hence $|x| \le a$ means $x>a$ and $-x<a$ $\Rightarrow$ $x>a$ and $x>-a$ or $-a<x<a$

2. $|x| > a$ means $-\infty.........-a..........0..............a.............\infty$: a<x<∝ and -∝<x<-a

Thus $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases} = \begin{cases} x, & -a<x<a \\ 0, & -\infty<x<-a \text{ and } a<x<\infty \end{cases}$

- **For evaluation of integrals we use inverse Fourier transform.**

Fourier Sine Transform:

$\Im_s[f(x)] = F_s(s) = \int_0^{\infty} f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$ and its inverse transform

$$\Im^{-1}{}_s[F(s)] = f_s(x) = \int_0^{\infty} F(s)\sin sx\,ds = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(s)\sin sx\,ds$$

Fourier Cosine Transform:

$\Im_c[f(x)] = F_c(s) = \int_0^{\infty} f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$ And its inverse transform

$$\Im^{-1}{}_c[F(s)] = f_c(x) = \int_0^{\infty} F(s)\cos sx\,ds = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(s)\cos sx\,ds$$

Laplace Transform:

Function ***F(t)***	**Laplace Traqnsform** ***L {F(t)} = f(p)***	**Function** ***L^{-1}{ f(p)}***	**Inverse Laplace Transform** ***F(t)***
L {F(t)} = f(p)	$\int_0^{\infty} e^{-pt}F(t)\,dt$ = *f(p)*		
L{1}	$\frac{1}{p}$, *p>0*	$L^{-1}\{\frac{1}{p}\}$, *p>0*	*1*

$L\{t\}$	$\frac{1}{p^2}$, $p>0$	$L^{-1}\{\frac{1}{p^2}\}$, $p>0$	t
$L\{t^n\}$	$\frac{n!}{p^{n+1}}$, $p>0$	$L^{-1}\{\frac{1}{p^{n+1}}\}$, $p>0$	$\frac{t^n}{n!}$
$L\{e^{at}\}$	$\frac{1}{p-a}$, $p>a$	$L^{-1}\{\frac{1}{p-a}\}$, $p>a$	e^{at}
$L\{\cos at\}$	$\frac{p}{p^2+a^2}$, $p>0$	$L^{-1}\{\frac{p}{p^2+a^2}\}$, $p>0$	$\cos at$
$L\{\sin at\}$	$\frac{a}{p^2+a^2}$, $p>0$	$L^{-1}\{\frac{1}{p^2+a^2}\}$, $p>0$	$\frac{\sin at}{a}$
$L\{\cosh at\}$	$\frac{p}{p^2-a^2}$, $\lvert p\rvert>0$	$L^{-1}\{\frac{p}{p^2-a^2}\}$, $\lvert p\rvert>0$	$\cosh at$
$L\{\sinh at\}$	$\frac{a}{p^2-a^2}$, $\lvert p\rvert>0$	$L^{-1}\{\frac{a}{p^2-a^2}\}$, $\lvert p\rvert>0$	$\frac{\sinh at}{a}$
Linear Property $L\{aF_1(t)+bF_2(t)\}$	$aL\{F_1(t)\}+bL\{F_2(t)\}$	***Linear Property*** $L^{-1}\{aF_1(t)+bF_2(t)\}$	$a L^{-1}F_1(t)+b L^{-1}F_2(t)$
First Shifting (Translation)Theorem $L\{e^{at} F(t)\}$	$f(p-a)$ where $f(p)= L\{F(t)\}$	***First Shifting (Translation)Theorem*** $L^{-1}\{f(p-a)\}$, where $f(p)= L\{F(t)\}$	$e^{at} L^{-1} F(t)\}$
Second Shifting *(Translation)* Theorem $L\{G(t)\}$, Where $G(t)=\begin{cases} F(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$	$e^{-ap} f(p)$ where $f(p)= L\{F(t)\}$	**Second Shifting *(Translation)* Theorem** $L^{-1}\{e^{-ap} f(p)\}$, where $f(p)= L\{F(t)\}$	$G(t)=\begin{cases} F(t-a) & ; t>a \\ 0 & ; t<a \end{cases}\}$
Change of Scale property $L\{F(at)\}$	$\frac{1}{a}f(\frac{p}{a})$, where $f(p)= L\{F(t)\}$	**Change of Scale property** $L^{-1}\{f(ap)\}$,	$\frac{1}{a}F(\frac{t}{a})$ where $F(t)= L^{-1}\{f(p)\}$
Differentiation Theorem $L\{F'(t)\}$	$p L\{F(t)\} - F(0)$		
$L\{F^n(t)\}$	$p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \ldots - F^{n-1}(0)$		
Integral Theorem If $F(t)$ is piecewise continuous function and $\lvert F(t)\rvert \le Me^{at}$ then $L\{\int_0^t F(x)dx\}$	$\frac{1}{p}L\{F(t)\}$		

Multiplication Theorem $L\{t\,F(t)\}$	$(-1)\frac{d}{dp}f(p)=-f'(p)$	***Multiplication Theorem*** $L^{-1}\{p\,f(p)\}$	$F'(t)$
$L\{t^n F(t)\}$	$(-1)\frac{d^n}{dp^n}f(p)$	$L^{-1}\{p^n \frac{d^n}{dp^n}f(p)\}$ $=L^{-1}\{p^n f^n(p)\}$	$F^n(t)=\frac{d^n}{dt^n}F(t)$ where $F(t)=L^{-1}\{f(p)\}$
Division Theorem $L\{\frac{F(t)}{t}\}$	$\int_p^\infty f(p)dp$	***Division Theorem*** $L^{-1}\{\frac{f(p)}{p}\}$	$F(t)=\int_0^t f(p)dp$
Fundamental theorem of periodic function *If F(t) is a periodic function of period T then L{F(t)}*	$\frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$	***Division Theorem*** $L^{-1}\{\frac{f(p)}{p^n}\}$	$F(t)=\int_0^t \ldots\ldots\ldots\ldots\int_0^t f(p)dp^n$

Convolution Theorem: If $L^{-1}\{f(p)\}=F(t)$ and $L^{-1}\{g(p)\}=G(t)$, where F and G are two function of Class A then

$$L^{-1}\{f(p).g(p)\}=\int_0^t F(x)G(t-x)dx=F*G$$

Heaviside's Expansion Theorem: If *f(p)* and *g(p)* are two polynomials in *p* ,where *degree f(p) <degree g(p)* . If g(p) is a polynomial of n- distinct zeros $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots\alpha_n$ then

$$L^{-1}\{\frac{f(p)}{g(p)}\}=\sum_{i=1}^{n}\frac{f(\alpha_i)}{g'(\alpha_i)}e^{\alpha_i t}=\frac{f(\alpha_1)}{g'(\alpha_1)}e^{\alpha_1 t}+\frac{f(\alpha_2)}{g'(\alpha_2)}e^{\alpha_2 t}+\ldots\ldots\ldots\ldots\frac{f(\alpha_i)}{g'(\alpha_i)}e^{\alpha_i t}$$

BASIC CONCEPTS OF PROBABILITY

EVENTS AND OUTCOMES

The result of an experiment is called an **outcome**. An **event** is any particular outcome or group of out comes.

A **simple event** is an event that cannot be broken down further. The **sample space** is the set of all possible simple events.

BASIC PROBABILITY

Given that all outcomes are equally likely, we can compute the probability of an event *E* using this formula:

$$\boxed{P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally - likely outcomes}}}$$

Cards: A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Complement of an Event: The complement of an event is the event "*E* doesn't happen".

The notation $\bar{E}$ is used for the complement of event E we can compute the probability of the complement using

$$\boxed{Q(E) = P\left(\bar{E}\right) = 1 - P(E)}$$

INDEPENDENT EVENTS : Events A and B are independent events if the probability of Event B occurring is the same whether or not Event A occurs *P*(*A* and *B*) for independent events

If events *A* and *B* are independent, then the probability of both *A* and *B* occurring is

P(A and B) = P(A) · P(B)

where *P*(*A* and *B*) is the probability of events *A* and *B* both occurring, *P*(*A*) is the probability of event *A* occurring, and *P*(*B*) is the probability of event *B* occurring *P*(*A* or *B*).

The probability of either *A* or *B* occurring (or both) is

P(A or B) = P(A) + P(B) – P(A and B)

CONDITIONAL PROBABILITY:

The probability the event *B* occurs, given that event *A* has happened, is represented as $P(B \mid A)$

This is read as "the probability of *B* given *A*" .

If Events *A* and *B* are not independent, then **P(A and B) = P(A) · P(B | A)**

BAYES' THEOREM :

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A})}$$

Experiment : An experiment is a test or series of tests in which purposeful changes are made to the input variables of a process or system so that we may observe and identify reasons for changes in the output response.

Random Experiment : A random experiment is one whose outcome cannot be predicted with certainty.

Random Variable : A random variable is a numerical description of the outcome of an experiment.

In other words : A random variable is a numerical variable whose measured value is determined by chance.

OR A random variable is a real valued function having domain as the sample space associated with a given random experiment.
Note: We will denote a random variable with an uppercase letter, such as X, and a measured value of the random variable with a lowercase letter, such as x.

TYPES OF RANDOM VARIABLE:

There are two common types of random variables. They are:
(a) **Discrete random variable:** a quantity assumes either a finite number of values or an infinite sequence of values, such as 0, 1, 2, …
or a variable, when real valued function defined on a discrete sample space is called a discrete random variable.
Example: The marks obtained in a paper , number of telephone calls per unit time , number of success in n-trails
(b) **Continuous random variable:** a quantity assumes any numerical value in an interval or collection of intervals,
Or A random variable X is said to be continuous if it takes aa possible values between certain limits. **Example**: time, weight, distance, and temperature.

PROBABILITY MASS FUNCTION:

Let X is a discrete random variable. A probability mass function (*p.m.f.*) is given by

(a) $P(X = a_i) = f(a_i) \ge 0,$ for every i.

(b) $\sum_{i=1}^{\infty} f(a_i) = f(a_1) + f(a_2) + \cdots + f(a_n) + \cdots = 1$

Or $f(x) = P\{x : X(x_i) = x\}$

Example : Let X be the number of heads , Then P.m.f.

Number of heads X={0,1}	Elementary events E	probability mass function (*p.m.f.*) *f(x)=P(X=x)*
x=0	T	1/2
x=1	H	1/2

Example: Toss a balanced coin twice. Let X be the number of heads . Find the probability mass function of X.

Solution.: Random variable $X=\{x=0,1,2\}$, Sample Space $S=\{HH,TT,HT,TH\}$ $n(S)=4$

Number of heads X={0,1,2}	Elementary events E	probability mass function (*p.m.f.*) *f(x)=P(X=x)*
x=0	TT	¼
x=1	HT,TH	2/4=1/2
X=2	HH	1/4

PROBABILITY DISTRIBUTION (OR DENSITY) FUNCTION (P.D.F)):

A function which describes, how probabilities are distributed over the values of the random variable is called distributive function.

i.e Let $f(x)$ is probability function then the probability density / probability distribution function is the function which represent the probabilities which lies in the given interval $[a,b]$, and defined by $P(a \le x \le b) = \int_a^b f(x)dx$.

Types of Distribution function:

(1) **Discrete probability distributions**: Discrete probability distributions are used when the sampling space is discrete but not countable. Following is a list of discrete probability distributions:

- discrete uniform
- binomial and multinomial
- hypergeometric
- negative binomial
- geometric
- Poisson

➢ **Required conditions for a discrete probability distribution function:**

Let $a_1, a_2, \ldots, a_n, \ldots$ be all the possible values of the *discrete* random variable *X*. Then, the required conditions for $f(x)$ to be the discrete probability distribution for *X* is

(a) $P(X = a_i) = f(a_i) \ge 0$, for every i. (b) $\sum_{i=1}^{\infty} f(a_i) = f(a_1) + f(a_2) + \cdots + f(a_n) + \cdots = 1$

(2) **Continuous probability distribution**: Continuous probability distribution is used when the sample space is continuous. Following is a list of continuous probability distributions:

- Uniform
- Normal (or Guassian)
- Gamma
- Beta
- t distribution
- F distribution
- χ^2 distribution

Required conditions for a continuous probability density:

Let the *continuous* random variable *Z* taking values in subsets of $(-\infty, \infty)$. Then, the required conditions for $f(x)$ to be the continuous probability density function for *Z* are

(a) $f(x) \geq 0, \quad -\infty < x < \infty.$ (b) $\int_{-\infty}^{\infty} f(x)dx = 1$

Example : Whether check the following function is p.d.f? $f(x) = 6x(1-x), 0 \leq x \leq 1$

Solution : Since $f(x) = 6x(1-x) \geq 0 \text{ in } 0 \leq x \leq 1$ and $\int_0^1 f(x)dx = \int_0^1 6x(1-x)dx = 1$, hence function isp.d.f..

Remark : for given distribution function p.d.f = $\boxed{f(x) = \frac{d}{dx}F(x)}$, where F(x)= distribution function.

MEAN / ARITHIMIC MEAN (OR EXPECTED VALUE) :

If X is discrete, $E(X) = \sum_{i=1}^{\infty} a_i f(a_i) = a_1 f(a_1) + a_2 f(a_2) + \cdots + a_n f(a_n) + \cdots$

If X is continuous, $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

VARIANCE:

If X is discrete, $\boxed{Var(X) = \sigma^2 = E[X - E(X)]^2 = \sum_i (a_i - \mu)^2 f(a_i)}$

$$= (a_1 - \mu)^2 f(a_1) + (a_2 - \mu)^2 f(a_2) + \cdots + (a_n - \mu)^2 f(a_n) + \cdots$$

If X is continuous $\boxed{Var(X) = \sigma^2 = E[X - E(X)]^2 = \int_{-\infty}^{\infty} (x-u)^2 f(x)dx}$

Note: In practice, it is easier to use the computational formula for the variance, rather than the defining formula:

$$\sigma^2 = E\left[X^2\right] - \mu^2 = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2 .$$

MOMENTS ABOUT ORIGIN:

(1) **First moment about origin = Mean :** $\mu_1' = \int_{-\infty}^{\infty} xf(x)dx$

(2) **r[th] moment about origin :** $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx$

(3) **First moment about mean =** $\mu_1 = 0$

(4) **Variance= Second moment about mean =** $\mu_2 = \mu_2' - (\mu_1')^2$

(5) **r[th] moment about mean :** $\int_a^1 3x^2 dx = 0.05$

(6) **Median:** Median is the line which divide the whole area under the curve in to two equal parts .

If M_d is median then $\boxed{\int_a^{m_d} f(x)dx = \frac{1}{2} \text{ or } \int_{m_d}^{b} f(x)dx = \frac{1}{2}}$

(7) **Mode : Mode is the value of x for which *f(x)* is maximum.**

(8) **Mean deviation from mean:** $M.D. = \int_a^b |x - \bar{x}| f(x)dx$

Remark: For symmetric distribution mean, mode and median coincides at origin. i.e. **mean=mode=median**

DISTRIBUTION FUNCTION or CUMULATIVE DISTRIBUTION FUNCTION (OR C.D.F.) :

1. Let X be a discrete random variable , then distributive function of X is
$\boxed{F(x) = P(X \le x) = \sum_{x_i \le x} p_i}$, such that $p_i \ge 0$ and $\sum_{i=1} p_i = 1$, where $p = p(x_i)$
2. Let X be continuous random variable then Cumulative distribution function is given by
$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx}$, for all $x \in \Re$.

If the distribution does not have a *p.d.f.*, we may still define the *c.d.f.* for any *x* as the probability that *X* takes on a value no greater than x.

Note: The *c.d.f.* for the distribution of a random variable is unique, and completely describes the distribution.

DISCRETE PROBABILITY DISTRIBUTION :

Example: A random variable *X* has the following probability function, find the distributive function

X=x	0	1	2	3
p(x)	0	1/5	2/5	2/5

Solution: $F(x) = P(X = x) = \sum_{x_i \le x} p_i$, $F(0) = P(X \le 0) = p(0) = 0$,

$F(1) = P(X \le 1) = p(0) + p(1) = 0 + 1/5 = 1/5$, $F(2) = P(X \le 2) = p(0) + p(1) + p(2) = 0 + 1/5 + 2/5 = 3/5$,

$F(3) = P(X \le 3) = p(0) + p(1) + p(2) + p(3) = 0 + 1/5 + 2/5 + 2/5 = 5/5 = 1$

PROBABILITY DISTRIBUTIONS

An example will make clear the relationship between random variables and probability distributions. Suppose you throw a coin two times. This simple statistical experiment can have four possible outcomes: HH, HT, TH, and TT. Now, let the variable *X* represent the number of Heads that result from this experiment. The variable *X* can take on the values 0, 1, or 2. In this example, *X* is **a random variable;** because its value is determined by the outcome of a statistical experiment.

A **probability distribution** is a table or an equation that links each outcome of a statistical experiment with its probability of occurrence. Consider the coin throw experiment described above. The table below, which associates each outcome with its probability, is an example of a probability distribution.

Number of heads	Probability
0	0.25
1	0.50
2	0.25

The above table represents the probability distribution of the random variable X.

CUMULATIVE PROBABILITY DISTRIBUTIONS:

A **cumulative probability** refers to the probability that the value of a random variable falls within a specified range.

Let us return to the coin throw experiment. If we throw a coin two times, we might ask: What is the probability that the coin throws would result in one or fewer heads? The answer would be a cumulative probability. It would be the probability that the coin throw experiment results in zero heads plus the probability that the experiment results in one head.

$$P(X < 1) = P(X = 0) + P(X = 1) = 0.25 + 0.50 = 0.75$$

Like a probability distribution, a cumulative probability distribution can be represented by a table or an equation. In the table below, the cumulative probability refers to the probability than the random variable X is less than or equal to x.

Number of heads: *x*	**Probability: *P(X = x)***	**Cumulative Probability: *P(X < x)***
0	0.25	0.25
1	0.50	0.75
2	0.25	1

UNIFORM PROBABILITY DISTRIBUTION:

The simplest probability distribution occurs when all of the values of a random variable occur with equal probability. This probability distribution is called the **uniform distribution**.

Uniform Distribution. Suppose the random variable X can assume k different values. Suppose also that the $P(X = x_k)$ is constant. Then,

$P(X = x_k) = 1/k$

Example: Suppose a die is tossed. What is the probability that the die will land on 6 ?

Solution: When a die is tossed, there are 6 possible outcomes represented by: $S = \{1, 2, 3, 4, 5, 6\}$. Each possible outcome is a random variable (X), and each outcome is equally likely to occur. Thus, we have a uniform distribution. Therefore, the $P(X = 6) = 1/6$.

Example: Suppose we repeat the dice tossing experiment described in Example 1. This time, we ask what is the probability that the die will land on a number that is smaller than 5?

Solution: When a die is tossed, there are 6 possible outcomes represented by: $S = \{1, 2, 3, 4, 5, 6\}$. Each possible outcome is equally likely to occur. Thus, we have a uniform distribution.

This problem involves a cumulative probability. The probability that the die will land on a number smaller than 5 is equal to:

$P(X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1/6 + 1/6 + 1/6 + 1/6 = 2/3$

Three main types of probability distributions are discussed in next section.

BINOMIAL DISTRIBUTION

To understand binomial distributions and binomial probability, it helps to understand binomial experiments and some associated notation; so we cover those topics first.

Binomial Experiment: A **binomial experiment** (also known as a **Bernoulli trial**) is a statistical experiment that has the following properties:

- The experiment consists of *n* repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by *P*, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

Consider the following statistical experiment. You throw a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We throw a coin 2 times.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant - 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

NOTATIONS:

The following notation is helpful, when we talk about binomial probability.

- *x*: The number of successes that result from the binomial experiment.
- *n*: The number of trials in the binomial experiment.
- *p*: The probability of success on an individual trial.
- *q*: The probability of failure on an individual trial. (This is equal to 1 - *p*.)
- b(*x*; *n, p*): Binomial probability - the probability that an *n*-trial binomial experiment results in exactly *x* successes, when the probability of success on an individual trial is *p*.
- ^nC_r: The number of combinations of *n* things, taken *r* at a time.

Binomial Distribution: Binomial distribution is a discrete probability distribution. A random variable is said to follow binomial distribution if it takes non negative values and its probability mass function is given by $P(x=r) = {}^nC_r p^r q^{n-r} = \dfrac{n!}{r!\ n-r!} p^r q^{n-r}$, $r = 0,1,2,3.....$

If an experiment is conducted in N- sets then , No. of *r*-Success in *n*- trails (or Frequency of success)=

$$N.P(x=r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!\ n-r!} p^r q^{n-r} \ , \ r = 0,1,2,3.....$$

Suppose we throw a coin two times and count the number of heads (successes). The binomial random variable is the number of heads, which can take on values of 0, 1, or 2. The binomial distribution is presented below.

Number of heads	Probability
0	0.25
1	0.50
2	0.25

The binomial distribution has the following properties:

The mean of the binomial distribution $\mu_{x=} np$.

The variance $\sigma_{2x=} n\,pq$.

The standard deviation $\sigma_x = \sqrt{n\,pq}$.

Binomial Probability: Suppose a binomial experiment consists of *n* trials and results in *x* successes. If the probability of success on an individual trial is P, then the binomial probability is:

$$b(x;\ n,\ P)\ =\ {}^nC_x\ p^x\ q^{n-x} = \frac{n!}{(n-x)!\,x!}p^x q^{n-x}\ for\ x=0,1,2,...,n$$

Or Probability of r – success in n-trails $P(X=r) = {}^nC_r\ p^r\ q^{n-r}$

Hypothesis of Binomial distribution :

1. The procedure has a **fixed number of trials**. [n trials]
2. The trials must be **independent**.
3. Each trial is in **one of two mutually exclusive categories**.
4. The **probabilities remain constant** for each trial.

POISSON distribution:

Definition: The **Poisson distribution** is a discrete probability distribution of a random variable *x* that satisfies the following conditions.

1. The experiment consists of counting the number of times, *x*, an event occurs in a given interval. The interval can be an interval of time, area, or volume.
2. The probability of two or more success in any sufficiently small subinterval is 0. For example, the fixed interval might be any time between 0 and 5 minutes. A subinterval could be any time between 1 and 2 minutes.
3. The probability of the event occurring is the same for any two intervals of equal length.
4. The number of occurrences (success) in any interval is independent of the number of occurrences in any other interval provided the intervals are not overlapping.

NECESSARY CONDITIONS FOR POISSON DISTRIBUTION:

Poisson distribution is a discrete probability distribution, which is the limiting case of the binomial distribution under certain conditions.

1. When n is very indefinitely very large
2. Probability of success is very small.
3. $np = \lambda$ is finite, $\lambda \in R^+$

Def: A discrete random variable X is said to be follow a Poisson distribution if the probability mass function is given by

$$p(X=x) = P(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,3.........\infty$$ Where e = 2.7183 and $\lambda > 0$

Here λ is called the ***parameter*** of the Poisson distribution.

Examples where the Poisson distribution is used (or) Applications of Poisson distribution:

This distribution is used to describe the behavior of the rare events like

1. The number of blind born per year in a large city.
2. The number of printing mistakes per page in a large volume of a book.
3. The number of air pockets in a glass sheet.
4. The number of accidents occurred annually at a busy crossing of city.
5. The number of defective articles produced by a quality machine.
6. This is widely used in waiting lines or queuing problems in management studies.
7. It has wide applications in industrial quality control.
8. In determining the number of deaths in a given period by a rare disease.

For a Poisson distribution the probability mass function is given by

$$\boxed{p(X=x)=P(x;\lambda)=\frac{e^{-\lambda}\lambda^{x}}{x!},x=0,1,2,3.........\infty}$$

Example: 1) Number of printing mistakes on each page of a book published by a good publisher

2) Number of telephone calls arriving at a telephone switch board per minute.

NORMAL DISTRIBUTION:

A random variable X is said to be normally distributed or to have a normal distribution if its p.d.f has the form

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty,\ -\infty < \mu < \infty, \text{ and } \sigma > 0.}$$

Here μ and σ are the parameters of the distribution; μ = the mean of the random variable X (or of the probability distribution); and σ = the standard deviation of X.

Note: The normal distribution is not just a single distribution, but rather a family of distributions; each member of the family is characterized by a particular pair of values of μ and σ.

The graph of the p.d.f. has the following characteristics:

1) It is a bell-shaped curve;
2) It is symmetric about μ;
3) The inflection points are at $\mu - \sigma$ and $\mu + \sigma$.

IMPORTANCE OF NORMAL DISTRIBUTION:

The normal distribution is very important in statistics for the following reasons:

1) Many phenomena occurring in nature or in industry have normal, or approximately normal, distributions.

Examples:

a) heights of people in the general population of adults;

b) for a particular species of pine tree in a forest, the trunk diameter at a point 3 feet above the ground;

c) fill weights of 12-oz. cans of Pepsi-Cola; d) IQ scores in the general population of adults;

e) diameters of metal shafts used in disk drive units.

2) Under general conditions (independence of members of a sample), the possible values of the sample mean for samples of a given (large) size have an approximate normal distribution (Central Limit Theorem).

THE EMPIRICAL RULE:

For the normal distribution,

(1) The probability that X will be found to have a value in the interval $(\mu - \sigma, \mu + \sigma)$ is approximately 0.6827;

(2) The probability that X will be found to have a value in the interval $(\mu - 2\sigma, \mu + 2\sigma)$ is approximately 0.9545;

(3) The probability that X will be found to have a value in the interval $(\mu - 3\sigma, \mu + 3\sigma)$ is approximately 0.9973.

Unfortunately, the p.d.f. of the normal distribution does not have a closed-form anti-derivative. Probabilities must be calculated using numerical integration methods. This difficulty is the reason for the importance of a particular member of the family of normal distributions, the standard normal distribution, which has *p.d.f.*

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \text{ for } -\infty < z < +\infty.$$

Note: For shorthand, we will write X ~ Normal(μ, σ) to mean that the continuous r.v. X has a normal distribution with mean μ and standard deviation σ.
The c.d.f. of the standard normal distribution will be denoted by

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} dw.$$

BETA DISTRIBUTION:

It is a continuous distribution.

- It is bounded on both sides. In this respect it resembles the binomial distribution. The standard beta distribution is constrained so that its domain is the interval (0, 1).
- The beta distribution has two parameters *a* and *b* both referred to as shape parameters.
- The formula for the beta density is the following.

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta} x^{\alpha-1}(1-x)^{\beta-1} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

The reciprocal of the ratio of gamma functions that appears in front as the normalizing constant is generally called the beta function and is denoted *B(α, β).*

- The beta distribution is often used in conjunction with the binomial distribution particularly in Bayesian models where it plays the role of a prior distribution for *p*.
- It also can be used to give rise to a beta-binomial model. Here the probability of success *p* is assumed to arise from a beta distribution and then, given the value of *p*, the observed number of successes has a binomial distribution with parameters *n* and this value of *p*. The significance of this approach is that it allows *p* to vary randomly between subjects and is a way of modeling what's called binomial over dispersion.

THE GAMMA DISTRIBUTION

Defn: The gamma function is defined by the integral $\Gamma(r) = \int_0^{+\infty} t^{r-1}e^{-t}dt$, for r > 0.

It may be shown using integration by parts that $\Gamma(r) = (r-1)\Gamma(r-1)$. Hence, in particular, if r is a positive integer, $\Gamma(r) = (r-1)!$. We also have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Defn: A continuous r.v. X is said to have a gamma distribution with parameters r > 0 and λ > 0 if the p.d.f. of X is $f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1}e^{-\lambda x}$, for x > 0, and f(x) = 0, for x ≤ 0.

The mean and variance of X are given by $\mu = E[X] = \frac{r}{\lambda}$ and $\sigma^2 = V(X) = \frac{r}{\lambda^2}$.

We write X ~ gamma(r, λ) to denote that X has a gamma distribution with parameters r and λ.

It may be easily shown that the integral of the gamma p.d.f. over the interval (0, +∞) is 1, using the definition of the gamma function.

The gamma distribution is very important in statistical inference, both in its own right and because it is the basis for constructing some other distributions useful in inference. For example, the "signal-to-noise" ratio statistic that we will use in analyzing the results of scientific experiments is based on a ratio of random variables which have gamma distributions of a particular form.

Defn: A continuous r.v. X is said to have a chi-squared distribution with k degrees of freedom if X ~ gamma(k, 0.5).

WEIBULL DISTRIBUTION:

Defn: A continuous r.v. X is said to have a Weibull distribution with parameters δ > 0 and β > 0 if the p.d.f. of X is

$f(x)=\frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$, for $x > 0$, and $f(x)=0$, for $x \le 0$. The mean and variance of X are $\mu=E[X]=\delta\Gamma\left(1+\frac{1}{\beta}\right)$ and $\sigma^2=V(X)=\delta^2\Gamma\left(1+\frac{2}{\beta}\right)-\mu^2$. We write X ~ Weibull(δ, β).

The c.d.f. for a Weibull(δ, β) distribution is given by $F(x)=1-\exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$, for $x > 0$, and

$F(x) = 0$, for $x \le 0$.
The Weibull distribution is used to model the reliability of many different types of physical systems. Different combinations of values of the two parameters lead to models with either a) increasing failure rates over time, b) decreasing failure rates over time, or c) constant failure rates over time.

THE UNIFORM DISTRIBUTION

Consider a continuous r.v. X whose distribution has p.d.f. $f(x)=\frac{1}{b-a}$, for $a \le x \le b$, and $f(x)=0$, otherwise. We say that X has a uniform distribution on the interval (a, b), abbreviated
X ~ Uniform(a, b). If we take a measurement of X, we are equally likely to obtain any value within the interval. Hence, for some subinterval $(c,d)\subseteq(a,b)$, we have $P(c \le x \le d)=\int_c^d \frac{1}{b-a}dx=\frac{d-c}{b-a}$.

The mean of the uniform distribution is $\mu=\int_{-\infty}^{+\infty} xf(x)dx=\int_a^b \frac{x}{b-a}dx=\frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b=\frac{a+b}{2}$, the midpoint of the interval (a, b).
The second moment of the distribution is

$$E\left[X^2\right]=\int_{-\infty}^{+\infty} x^2 f(x)dx=\frac{1}{b-a}\int_a^b x^2dx=\frac{b^3-a^3}{3(b-a)}=\frac{(b-a)(b^2+ab+a^2)}{3(b-a)}.$$

Then the variance is

$\sigma^2=E\left[X^2\right]-\mu^2=\frac{b^2+ab+a^2}{3}-\frac{b^2-2ab+a^2}{4}=\frac{(b-a)^2}{12}$, and the standard deviation is $\sigma=\frac{b-a}{2\sqrt{3}}$.

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